First examples of non-abelian quotients of the Grothendieck-Teichmueller group

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If a field extension $E \supset F$ satisfies some nice properties then the corresponding automorphism group $Aut(E/F)$ (automorphisms of E that fixing F) is called the **Galois group** $Gal(E/F)$.

Definition (*Absolute Galois group $G_{\mathbb{Q}}$*)

If we consider an infinite extension $\overline{\mathbb{Q}} \supset \mathbb{Q}$, then the corresponding Galois group is called the absolute Galois group of rational numbers and denoted $G_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Although this group is uncountable, we know explicitly only two elements:

- Identity
- Complex conjugation ($a + bi \to a - bi$)

It plays an important role in number theory and is one of the most mysterious objects in mathematics!

In 1990, Vladimir Drinfield introduced yet another mysterious group $\widehat{GT}$ (which he called the Grothendieck-Teichmueller group). $\widehat{GT}$ is a subset of

$$\widehat{\mathbb{Z}} \times \widehat{F_2}$$

($\widehat{\mathbb{Z}}$ and $\widehat{F_2}$ are profinite completion of $\mathbb{Z}$ and F_2 respectively) where all pairs satisfy two hexagon relations and the pentagon relation.
The group $\widehat{GT}$ receives a homomorphism

$$Ih : G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$$

It was shown that *Ih* is injective and we call it the **Ihara embedding.**
In his 1990 ICM address, Yasutaka Ihara asked: Is this homomorphism surjective?.
This question is still open and **VERY HARD!!!**

There are also a lot of open questions about the structure of $\widehat{GT}$. For example, here is one from Loshack and Schneps paper called ”Open problems in Grothendieck-Teichmueller Theory”:

Question

Can anything be said about the finite quotients of $\widehat{GT}$? Obviously, all abelian groups arise as quotients, since $\widehat{\mathbb{Z}}^{\times}$ is a quotient of $\widehat{GT}$. But what non-abelian groups arise?

In our paper, we found the first examples of the non-abelian finite quotients of $\widehat{GT}$. We did it by producing versions of Ih that are surjective.

Surprisingly, in our proof, we used relatively elementary tools:

- Basic properties of group homomorphisms;
- The surjectivity of the cyclotomic character $\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^{\times}$;
- The image of complex conjugation in $\widehat{GT}$ is $(-1_{\widehat{\mathbb{Z}}}, 1_{\widehat{F_2}})$;
- Basic number theory

What is the cyclotomic character?

Let $\mathbb{Q}^{ab}$ be a subfield of $\overline{\mathbb{Q}}$ that is obtained by extending $\mathbb{Q}$ with all roots of unity, one can show that $\mathbb{Q}^{ab}$ is a Galois extension of $\mathbb{Q}$. Also $Gal(\mathbb{Q}^{ab}/\mathbb{Q})$ is isomorphic to the group $\widehat{\mathbb{Z}}^{\times}$ of units of the ring $\widehat{\mathbb{Z}}$.

There is a natural surjective homomorphism (restriction) $G_{\mathbb{Q}} \to Gal(\mathbb{Q}^{ab}/\mathbb{Q})$. So we get a surjective group homomorphism

$$\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^{\times}$$

and we call it **cyclotomic character.**

Actually, for $g \in G_{\mathbb{Q}}$:

$$Ih(g) = \left(\frac{\chi(g) - 1}{2}, f_g \right) \in \widehat{\mathbb{Z}} \times \widehat{F_2}$$

A bit about braid groups

Let B_3 be the **Artin braid group** on 3 strands:

$$B_3 := \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

The generators of B_3 can be represented by these pictures:

Let PB_3 be the corresponding pure braid group, i.e PB_3 is the kernel of the standard homomorphism $B_3 \to S_3$:

$$\sigma_1 \mapsto (1,2), \quad \sigma_2 \mapsto (2,3)$$

For our purposes, we need the following poset:

$$NFI_{PB_3}(B_3) := \{K \mid K \trianglelefteq B_3, \mid K \leq PB_3, \mid |B_3 : K| < \infty\}.$$

The groupoid $GTSh$ of GT-shadows

The group $\widehat{GT}$ is closely connected to the groupoid $GTSh$ with

$$Ob(GTSh) := NFI_{PB_3}(B_3).$$

For $K, N \in NFI_{PB_3}(B_3)$, the set of morphisms $GTSh(K, N)$ is a subset of pairs

$$(m + N_{ord}\mathbb{Z}, fN) \in \mathbb{Z}/N_{ord}\mathbb{Z} \times PB_3/N$$

satisfying certain conditions. Here N_{ord} is a positive integer that can be expressed in terms of N. We call such pairs **GT-shadows** with the target N.

These conditions are needed to guarantee that the formulas

$$T_{m,f}(\sigma_1) := \sigma_1^{2m+1}N, \qquad T_{m,f}(\sigma_2) := f^{-1}\sigma_2^{2m+1}fN,$$

defined a group homomorphism $T_{m,f} : B_3 \to B_3/N$ and $ker(T_{m,f}) = K$.

Ihara homomorphism for $GTSh$

If $K \in Ob(GTSh)$ is **isolated**, then the connected component $GTSh_{conn}(K)$ is basically the finite group

$$GTSh(K, K).$$

We will denote $GTSh(K, K) = GT(K)$.For every isolated object we have a natural group homomorphism (projection map)

$$PR_K : \widehat{GT} \to GT(K).$$

Composing it with Ih we get

$$Ih_K : G_{\mathbb{Q}} \to GT(K).$$

Isolated objects we explored

We looked at the subposet

$$Dih := \{K^{(n)} \in NFI_{PB_3}(B_3) | n \in \mathbb{Z}_{\geq 3}\} \subset NFI_{PB_3}(B_3)$$

related to dihedral groups. More precisely,

$$K^{(n)} := ker(\psi_n : PB_3 \to D_n \times D_n \times D_n)$$

$$\psi_n(\sigma_1^2) := (r, s, s), \;\; \psi_n(\sigma_2^2) := (rs, r, rs), \;\; \psi_n((\sigma_1\sigma_2\sigma_1)^2) := (1, 1, 1).$$

These three elements generate PB_3.

Theorem (*I.B., B.H., V.P., V.D.*)

If $n = 2^a$ for $a \geq 2$, then

$$Ih_{K^{(n)}} : G_{\mathbb{Q}} \to GT(K^{(n)})$$

is surjective.

Future directions

- Explore action of *GTSh* on child drawings;
- Develop a SageMath package for working with *GT*-shadows;
- Inverse Galois theory;
- Describe extensions of $\mathbb{Q}$ that correspond to $GT(K^{(n)})$

Selected References

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THANK YOU!